Chapter 4
Asymmetric Cryptography

- Introduction
- Encryption: RSA
- Key Exchange: Diffie-Hellman

Asymmetric Cryptography

- General idea:
  - Use two different keys $-K$ and $+K$ for encryption and decryption
  - Given a random ciphertext $c = E(+K, m)$ and $+K$ it should be infeasible to compute $m = D(-K, c) = D(-K, E(+K, m))$
    - This implies that it should be infeasible to compute $-K$ when given $+K$
  - The key $-K$ is only known to one entity A and is called A's ***private key*** $\boldsymbol{-K_A}$
  - The key $+K$ can be publicly announced and is called A's ***public key*** $\boldsymbol{+K_A}$
- Applications:
  - ***Encryption:*** If B encrypts a message with A's public key $+K_A$, he can be sure that only A can decrypt it using $-K_A$
  - ***Signing:*** If A encrypts a message with his own private key $-K_A$, everyone can verify this signature by decrypting it with A's public key $+K_A$
- Attention: It is crucial, that everyone can verify that he really knows A's public key and not the key of an adversary!

Design of Asymmetric Cryptosystems

- Difficulty: Find an ***encryption algorithm*** and a ***key generating method*** to construct two keys $-K$, $+K$ such that it is not possible to decipher $E(+K, m)$ with the knowledge of $+K$
  - Constraints:
    - The key length should be "manageable"
    - Encrypted messages should not be arbitrarily longer than unencrypted messages (we would tolerate a small constant factor)
    - Encryption and decryption should not consume too much resources (time, memory)
  - Basic idea: Take a problem in the area of mathematics / computer science, that is *hard* to solve when knowing only $+K$, but *easy* to solve when knowing $-K$
    - Knapsack problems: basis of first working algorithms, which were unfortunately almost all proven to be insecure
    - Factorization problem: basis of the ***RSA algorithm***
    - Discrete logarithm problem: basis of ***Diffie-Hellman*** and ElGamal

RSA – Mathematical Background (Modular Arithmetic)

- We say *b is congruent a mod n* if it has the same remainder like *a* when divided by *n*. So, *n* divides *(a-b)*, and we write $\boldsymbol{b \equiv a \bmod n}$
  - E.g., $4 \equiv 11 \bmod 7$, $25 \equiv 11 \bmod 7$
- Greatest common divisor
  - Let $a, b \in Z$ and $d = \gcd(a, b)$. Then there exists $m, n \in Z$ such that: $\boldsymbol{d = m \times a + n \times b}$
- **Euler totient of *n*: $\Phi(n)$**
  - Let $\Phi(n)$ denote the number of positive integers less than *n* and relatively prime to *n*
    - Examples: $\Phi(4) = 2$, $\Phi(15) = 8$
    - If *p* is prime $\Rightarrow \Phi(p) = p - 1$
  - Let *n* and *b* be positive and relatively prime integers, i.e. $gcd(n, b) = 1$ $\Rightarrow \boldsymbol{b^{\Phi(n)} \equiv 1 \bmod n}$
- Euclidean Algorithm
  - The algorithm *Euclid* given *a, b* computes gcd(*a, b*)
  - *int Euclid(int a, b) {*
    if (b = 0) { return(a);}
    return(Euclid(b, a MOD *b);*
    }
- **Extended Euclidean Algorithm**
  - The algorithm *ExtEuclid* given *a, b* computes *d, m, n* such that: $d = gcd(a, b) = m \times a + n \times b$
  - *struct{int d, m, n} ExtEuclid(int a, b) {*
    int d, d', m, m', n, n';
    if (b = 0) {return(a, 1, 0); }
    (d', m', n') = ExtEuclid(b, a MOD *b);*
    $(d, m, n) = (d', n', m' - \lfloor a / b \rfloor \times n');$
    return(d, m, n);
    }

For more information, please refer to undergraduate CS classes or to the NetSec slides WS 2006/2007

RSA in a Nutshell

- Invented by R. Rivest, A. Shamir and L. Adleman [RSA78]

- ***Key generation***
  - Select p, q — p and q both prime, $p \neq q$
  - Calculate n — $n = p \times q$
  - Calculate $\Phi(n)$ — $\Phi(n) = (p - 1)(q - 1)$
  - Select integer e — $\gcd(\Phi(n), e) = 1$; $1 < e < \Phi(n)$
  - Calculate d — $d \times e \bmod \Phi(n) = 1$ (extended Euclid)
  - Public key — $+K = \{e, n\}$
  - Private key — $-K = \{d, n\}$

- ***Encryption***
  - Plaintext — $M < n$ (what about 0 , 1, …?)
  - Ciphertext — $C = M^e \bmod n$

- ***Decryption***
  - Ciphertext — C
  - Plaintext — $M = C^d \bmod n$

RSA – Encryption / Decryption

- Let p, q be distinct large primes and $n = p \times q$. Assume, we have also two integers e and d such that $d \times e \equiv 1 \bmod \Phi(n)$
- Let M be an integer that represents the message to be encrypted, with M positive, smaller than and relatively prime to n.
  - Example: Encode with <blank> = 99, A = 10, B = 11, ..., Z = 35
    So “HELLO” would be encoded as 1714212124.
    If necessary, break M into blocks of smaller messages: 17142 12124

- To encrypt, compute: $C = M^e$ MOD n
  - This can be done efficiently using the *square-and-multiply algorithm*
- To decrypt, compute: $M' = C^d$ MOD n

- Proof

  $d \times e \equiv 1 \bmod \Phi(n) \Rightarrow \exists\, k \in Z: (d \times e) - 1 = k \times \Phi(n) \Leftrightarrow (d \times e) = k \times \Phi(n) + 1$

  we have: $M' \equiv E^d \equiv M^{(e \times d)} \equiv M^{(k \times \Phi(n) + 1)} \equiv 1^k \times M \equiv M \bmod n$ ■

RSA – Encryption / Decryption

- As $(d \times e) = (e \times d)$ the operation also works in the opposite direction, that means you can encrypt with d and decrypt with e

- This property allows to use the same keys d and e for:
  - Receiving messages that have been encrypted with one’s public key
  - Sending messages that have been signed with one’s private key

RSA – Security

- The security of the scheme lies in the difficulty of factoring $n = p \times q$ as it is easy to compute $\Phi(n)$ and then d, when p and q are known

- This class will not teach why it is difficult to factor large n’s, as this would require to dive deep into mathematics
  - If p and q fulfill certain properties, the best known algorithms are exponential in the number of digits of n
    - Please be aware that if you choose p and q in an “unfortunate” way, there might be algorithms that can factor more efficiently and your RSA encryption is not at all secure:
      - Thus, p and q should be about the same bit length and sufficiently large
      - $(p - q)$ should not be too small
      - If you want to choose a small encryption exponent, e.g. 3, there might be additional constraints, e.g. $\gcd(p - 1, 3) = 1$ and $\gcd(q - 1, 3) = 1$
    - The security of RSA also depends on the primes generated being truly random (like every key creation method for any algorithm)
  - Moral: If you are to implement RSA by yourself, ask a mathematician or better a cryptographer to check your design

RSA - Security

- Side channel attacks
  - Optimizations for use of RSA in embedded systems depend on the Chinese remainder theorem (CRT)
    - Applications
      - Smart cards (token, banking)
      - Pay-per-view TV
      - and many others…
    - Use (and storage) of p and q allows to calculate m^e mod p, which can be efficiently manipulated to compute m^e mod n
    - Introducing computation errors allows to reveal the prime p
      p = gcd(s'-s,n) with s' and s being the bogous and correct signatures
  - Implementation using square and multiply
    - Most RSA implementations rely on the square-and-multiply algorithm for the exponentiations
    - Timing attacks can by used to "guess" the private key

[A. G. Voyiatzis, "An Introduction to Side Channel Cryptanalysis of RSA", ACM Crossroads, vol. 11.3, 2004]

Diffie-Hellman – Mathematical Background

- ***Finite groups***
  - Abelian group: set *S* and a binary and a binary operation ⊕: (*S*, ⊕), with the following properties: closure, identity, associativity, commutativity and inverse elements
  - Finite group: Abelian group plus finite set of elements , i.e. $|S| < \infty$
- ***Primitive root, generator***
  - Let (S, •) be a group, $g \in S$ and $g^a := g \bullet g \bullet \ldots \bullet g$ (*a* times with $a \in Z^+$)
    Then *g* is called a *primitive root* of (S, •) $:\Leftrightarrow \{g^a \mid 1 \leq a \leq |S|\} = S$
  - Examples:
    - 1 is a primitive root of $(Z_n, +_n)$
    - 3 is a primitive root of $(Z^*_7, \times_7)$
  - $(Z^*_n, \times_n)$ does have a primitive root $\Leftrightarrow n \in \{2, 4, p, 2 \times p^e\}$ where *p* is an odd prime and $e \in Z^+$

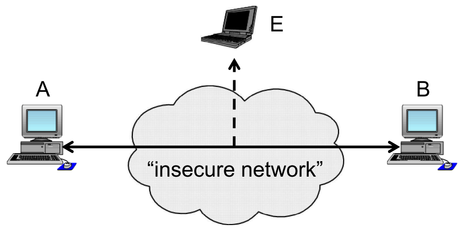
Diffie-Hellman – Mathematical Background

- Definition: ***discrete logarithm***
  - Let *p* be prime, *g* be a primitive root of $(Z^*_p, \times_p)$ and *c* be any element of Z^*_p. Then there exists *z* such that: $\boldsymbol{g^z \equiv c \bmod p}$
    z is called the *discrete logarithm* of *c* modulo *p* to the base *g*
  - Example:
    - 6 is the discrete logarithm of 1 modulo 7 to the base 3 as $3^6 \equiv 1 \bmod 7$
  - The calculation of the discrete logarithm *z* when given *g, c,* and *p* is a computationally difficult problem and the asymptotical runtime of the best known algorithms for this problem is exponential in the bit length of *p*

For more information, please refer to undergraduate CS classes or to the NetSec slides WS 2006/2007

Diffie-Hellman Key Exchange

- The Diffie-Hellman key exchange was first published in the landmark paper [DH76], which also introduced the fundamental idea of asymmetric cryptography
- The DH exchange in its basic form enables two parties A and B to agree upon a *shared secret* using a public channel:
  - *Public channel* means, that a potential attacker E (E stands for eavesdropper) can ***read*** all messages exchanged between A and B



Key Exchange Procedure

- A chooses a prime p, a primitive root g of Z^*_p, and a random number q:
  - A and B can agree upon the values p and g prior to any communication, or A can choose p and g and send them with his first message
  - A computes $v = g^q$ MOD p and sends to B: $\{p, g, v\}$
- B chooses a random number r:
  - B computes $w = g^r$ MOD p and sends to A: $\{p, g, w\}$ (or just $\{w\}$)
- Both sides compute the common secret:
  - A computes $s = w^q$ MOD p
  - B computes $s' = v^r$ MOD p
- As $g^{(q \times r)}$ MOD $p = g^{(r \times q)}$ MOD p it holds: $s = s'$
- Remark: In practice the number g does not necessarily need to be a primitive root of p, it is sufficient if it generates a large subgroup of Z^*_p

Diffie-Hellman Key Exchange

- The mathematical basis for the DH exchange is the problem of finding *discrete logarithms in finite fields*
  - An attacker Eve (E) who is listening to the public channel can only compute the secret s, if she is able to compute either q or r which are the discrete logarithms of v, w modulo p to the base g
- It is important, that A and B can be sure, that the attacker is not able to ***alter*** messages, as in this case he might launch a *man-in-the-middle attack*
- Remark: The DH exchange is *not* an asymmetric encryption algorithm, but is nevertheless introduced here as it goes well with the mathematical flavor of this lecture... :o)

Diffie-Hellman Key Exchange – Man-in-the-middle attack

- Eve generates to random numbers q' and r':
  - Eve computes $v' = g^{q'}$ MOD p and $w' = g^{r'}$ MOD p
- When A sends $\{p, g, v\}$ she intercepts the message
  - Then, E sends to B: $\{p, g, v'\}$
- When B sends $\{p, g, w\}$ she intercepts the message as well
  - E sends to A: $\{p, g, w'\}$
- When the supposed "shared secret" is computed we get:
  - A computes $s_1 = w'^q$ MOD p = $v^{r'}$ MOD p the latter computed by E
  - B computes $s_2 = v'^r$ MOD p = $w^{q'}$ MOD p the latter computed by E
  - So, in fact A and E have agreed upon a shared secret s_1, similarly E and B have agreed upon a shared secret s_2
- E can now use the "shared secret" to intercept all the messages encrypted by this key to forge and re-encrypt the messages without being noticed

Diffie-Hellman Key Exchange

- Two countermeasures against the man-in-the-middle attack:
  - The shared secret is *"authenticated"* after it has been agreed upon
    - We will treat this in the section on key management
  - A and B use a so-called *interlock protocol* after agreeing on a shared secret:
    - For this they have to exchange messages that E has to relay before she can decrypt / re-encrypt them
    - The content of these messages has to be checkable by A and B
    - This forces E to invent messages and she can be detected
    - One technique to prevent E from decrypting the messages is to split them into two parts and to send the second part before the first one.
      - If the encryption algorithm used inhibits certain characteristics E can not encrypt the second part before she receives the first one.
      - As A will only send the first part after he received an answer (the second part of it) from B, E is forced to invent two messages, before she can get the first parts.

Conclusion

- Asymmetric cryptography allows to use two different keys for:
  - Encryption / Decryption
  - Signing / Verifying
- The most practical algorithms that are still considered to be secure are:
  - RSA, based on the difficulty of factoring
  - Diffie-Hellman (not an asymmetric algorithm, but a key agreement protocol)
  - ElGamal, like DH based on the difficulty of computing discrete logarithms
- As their security is entirely based on the difficulty of certain mathematical problems, algorithmic advances constitute their biggest threat

- Practical considerations:
  - Asymmetric cryptographic operations are magnitudes slower than symmetric ones
  - Therefore, they are often not used for encrypting / signing bulk data
  - Symmetric techniques are used to encrypt / compute a cryptographic hash value and asymmetric cryptography is just used to encrypt a key / hash value

Summary (what do I need to know)

- Principles of asymmetric cryptography
  - +K, -K for encryption and signing
  - Mathematical problems that are hard to solve
  - Factorization, discrete logarithm

- RSA
  - Key generation
  - Encryption / decryption (how?, why does it work?)

- Diffie-Hellman key exchange
  - Key generation procedure
  - Man-in-the-middle attack

Additional References

[Bre88a] D. M. Bressoud. *Factorization and Primality Testing.* Springer, 1988.

[Cor90a] T. H. Cormen, C. E. Leiserson, R. L. Rivest. *Introduction to Algorithms.* The MIT Press, 1990.

[DH76] W. Diffie, M. E. Hellman. *New Directions in Cryptography.* IEEE Transactions on Information Theory, IT-22 , pp. 644-654, 1976.

[ElG85a] T. ElGamal. *A Public Key Cryptosystem and a Signature Scheme based on Discrete Logarithms.* IEEE Transactions on Information Theory, Vol.31, Nr.4, pp. 469-472, July 1985.

[Kob87a] N. Koblitz. *A Course in Number Theory and Cryptography.* Springer, 1987.

[Men93a] A. J. Menezes. *Elliptic Curve Public Key Cryptosystems.* Kluwer Academic Publishers, 1993.

[Niv80a] I. Niven, H. Zuckerman. *An Introduction to the Theory of Numbers.* John Wiley & Sons, 4th edition, 1980.

[RSA78] R. Rivest, A. Shamir und L. Adleman. *A Method for Obtaining Digital Signatures and Public Key Cryptosystems.* Communications of the ACM, February 1978.